<u>From Newton's Law of Gravitation to Kepler's First Law and Connecting Kepler's 2nd Law toConservation of Angular Momentum</u>

When a planet *P* along a curve has polar coordinates (r, θ), the vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are defined by

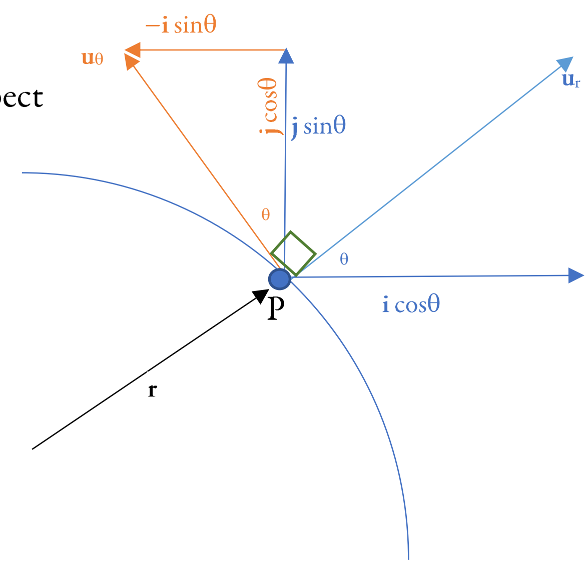
$\mathbf{u}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ and

$\mathbf{u}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$,

where **i** and **j** are unit vectors in the directions of the positive *x*- and *y*-axes. Then $\mathbf{u}_r$ is a unit vector along *OP(where O is the focal point of curve)* in the direction of increasing *r*, and $\mathbf{u}_\theta$ is a unit vector perpendicular to this in the direction of increasing θ.

Any vector, such as acceleration, ***a*,** can be written in terms of its components in the directions of $\mathbf{u}_r$ and $\mathbf{u}_\theta$. Thus $\boldsymbol{a} = a_r\mathbf{u}_r + a_\theta\mathbf{u}_\theta$, ...eq (1)

where the component a_r is a scalar and a radial component of acceleration, and the component a_θ is the scalar transverse component.



The rate of change of $\mathbf{u}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$, with respect to time, t, is:

$$\frac{(d\boldsymbol{u_r})}{dt} = \frac{d}{dt}(\mathbf{i}\cos\Theta + \mathbf{j}\sin\Theta)$$

=

$$\frac{(d\boldsymbol{u_r})}{dt} = \frac{d}{dt}\left(\mathbf{i}\cos\Theta\frac{d\Theta}{d\Theta} + \mathbf{j}\sin\Theta\frac{d\Theta}{d\Theta}\right)$$

$$\frac{(d\boldsymbol{u_r})}{dt} = \frac{d}{d\Theta}\left(\mathbf{i}\cos\Theta\frac{d\Theta}{dt} + \mathbf{j}\sin\Theta\frac{d\Theta}{dt}\right)$$

$$\frac{(d\boldsymbol{u_r})}{dt} = \left(-\mathbf{i}\sin\Theta\frac{d\Theta}{dt} + \mathbf{j}\cos\Theta\frac{d\Theta}{dt}\right) = (-\mathbf{i}\sin\Theta + \mathbf{j}\cos\Theta)\frac{d\Theta}{dt} = \mathbf{u}_\theta\frac{d\Theta}{dt}$$

Or in shorthand: $\boldsymbol{u_r}' = \mathbf{u}_\theta\,\Theta'$ ...(eq 2)

The rate of change of $\mathbf{u}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$, with respect to time, t, is:

$$\frac{(d\boldsymbol{u}_\Theta)}{dt} = \frac{d}{dt}(-\mathbf{i}\sin\Theta + \mathbf{j}\cos\Theta) = \frac{d}{dt}\left(-\mathbf{i}\sin\Theta\frac{d\Theta}{d\Theta} + \mathbf{j}\cos\Theta\frac{d\Theta}{d\Theta}\right)$$

$$=\frac{d}{d\Theta}\left(-\mathbf{i}\sin\Theta\frac{d\Theta}{dt} + \mathbf{j}\cos\Theta\frac{d\Theta}{dt}\right) = (-\mathbf{i}\cos\Theta - \mathbf{j}\sin\Theta)\frac{d\Theta}{dt} = -u_r\frac{d\Theta}{dt}$$

Or in shorthand: $\boldsymbol{u}_\Theta{}' = -u_r\Theta'$ ...(eq 3)

The velocity of the planet, ***v***, is the rate of change of the radial vector, ***r***, with respect to time. The radial vector, ***r***, in turn equals ru_r.

So $\boldsymbol{v} = \frac{dr}{dt} = \frac{d}{dt}(\mathrm{r}u_r) = \mathrm{r}'u_r + \mathrm{r}u_r{}'$.

Substituting eq (2) int eq (4):

$$\boldsymbol{v} = \frac{dr}{dt} = r'u_r + \boldsymbol{r}\boldsymbol{u}_\Theta\,\Theta'$$

Acceleration is the derivative of velocity with respect to time:

$$\boldsymbol{a} = \frac{d^2r}{dt^2} = \mathrm{r}''u_r + \mathrm{r}'u_r{}' + \mathbf{r}'\mathbf{u}_\Theta\,\Theta' + \mathbf{r}(\mathbf{u}_\Theta\,\Theta')' =$$

$$\mathrm{r}''u_r + \mathrm{r}'u_r{}' + \mathbf{r}'\boldsymbol{u}_\Theta\,\Theta' + \mathbf{r}(\boldsymbol{u}_\Theta'\Theta' + \mathbf{u}_\Theta\,\Theta'')$$ Substituting equations (2) and (3)

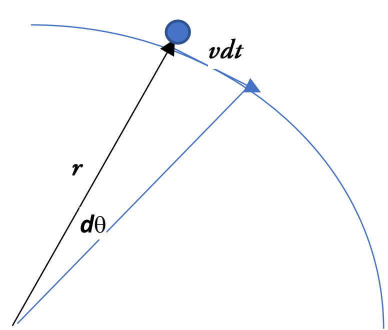
$$= \mathrm{r}''u_r + \mathbf{r}'\boldsymbol{u}_\Theta\,\Theta' + \mathbf{r}'\boldsymbol{u}_\Theta\,\Theta' + r(-u_r\Theta'\Theta' + \mathbf{u}_\Theta\,\Theta'')$$

$$\boldsymbol{a} = u_r(\mathrm{r}'' - \mathrm{r}(\Theta')^2) + \boldsymbol{u}_\Theta\,(2\mathrm{r}'\Theta' + \mathrm{r}\Theta'')$$ ... eq (4)

Recall equation (1): $\boldsymbol{a} = a_r\mathbf{u}_r + a_\Theta\mathbf{u}_\Theta$,

So $a_r = \mathrm{r}'' - \mathrm{r}(\Theta')^2$ *and* $a_\Theta = 2\mathrm{r}'\Theta' + \mathrm{r}\Theta''$ ...eq (5a) and (5b)

<u>Conservation of Angular Momentum</u>



The area of the triangular region is ½ the cross product of *r* and *vdt* or $\mathrm{dA} = \frac{r \times vdt}{2}$

$\frac{dA}{dt} = \frac{r \times v}{2}$ Multiply both sides by the mass of the planet,

$m\frac{dA}{dt} = \frac{r \times m\, v}{2} = \frac{L}{2}$ where L is the angular momentum.

$$\frac{dA}{dt} = \frac{L}{2m}$$ ....eq(6)

From Kepler's 2nd Law which observed that planets sweep equal areas in equal times, we know that $\frac{L}{2m}$ is a constant. In addition, if the force between a planet and its star is a central force, F , then the torque, t, which is the cross product between r and F will equal zero.

Since F is the change in momentum, p with respect to time,

$$t = r \times F = 0 = r \times \frac{dp}{dt} = r \times \frac{mv}{dt} = \frac{dL}{dt} = 0$$

If the rate of change of momentum is zero then L must be a constant, and L is conserved.

In eq(6), let the constant h = L/m.

$\frac{dA}{dt} = \frac{h}{2}$ ...eq (7)

For polar equations, where r is a function of θ,

$dA = \frac{r^2 d\theta}{2}$ Dividing both sides by *dt* and then comparing to eq (7)

$\frac{dA}{dt} = \frac{r^2 d\theta}{2\, dt} = \frac{h}{2}$. Solving for $\frac{d\theta}{dt} = \frac{h}{r^2}$ ...eq (8)

Since $r^2 \frac{d\theta}{dt} = h = constant$, if we differentiate implicitly,

$(r^2\theta')' = h' = 0$

$2rr'\theta' + r^2\theta'' = 0$...dividing through by r:

$2r'\theta' + r\theta'' = 0$. ...eq (9)

Returning to eq (5a) and (5b):

$a_r = r'' - r(\Theta')^2$ *and* $a_\Theta = 2r'\Theta' + r\Theta''$ eq (5a) and (5b)

By comparing eq 5(b) to eq 9, we notice that $a_\theta = 0$, and since $\boldsymbol{a} = a_r\mathbf{u}_r + a_\theta\mathbf{u}_\theta$, eq(1),

$a = a_r u_r = (r'' - r(\Theta')^2)u_r$

According to Newton's gravitational law, the central force is inversely proportional to the square of the radius and directly proportional to the product of the mass of the planet, m, and the mass of the sun, M. It acts towards the center and is therefore in the direction opposite to the radial vector u_r.

$F = -\frac{GMm}{r^2}u_r$. By the law of inertia $F = m\boldsymbol{a} = \boldsymbol{m} a_r\, u_r$

Equating the above two we get:

$$-\frac{GMm}{r^2}u_r = \boldsymbol{m}a_r\, u_r$$

$$\text{Or } -\frac{GM}{r^2} = a_r \qquad \text{...eq (10)}$$

Substituting eq (5a) $a_r = r'' - r(\Theta')^2$ into eq 10:

$$r'' - r(\Theta')^2 = -\frac{GM}{r^2} \qquad \text{...eq(11)}$$

To solve the above differential equation, we choose a function $w(\Theta)$, such that $w = 1/r$ or $r = 1/w$ ... eq(12)

$$w' = \frac{dw}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{dr}{dr} = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{d\theta} = \left(\frac{-1}{r^2}\right)\frac{dr}{dt}\frac{dt}{d\theta}$$

$w' = \left(\frac{-1}{r^2}\right)\frac{dr}{dt}\frac{dt}{d\theta}$ and solving for $\frac{dr}{dt}$,

$\frac{dr}{dt} = -r^2 w' \frac{d\theta}{dt}$, but from eq (8), $\frac{d\theta}{dt} = \frac{h}{r^2}$

$$\frac{dr}{dt} = -r^2 w' \frac{h}{r^2} = -hw'$$

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{dw}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{dw}{dt}\right) = -h\frac{d}{d\theta}\left(\frac{dw}{d\theta}\right)\left(\frac{d\theta}{dt}\right)$$

$$\frac{d^2r}{dt^2} = r'' = -h\frac{d^2w}{d\theta^2}\left(\frac{d\theta}{dt}\right) = -hw''\frac{h}{r^2} = -h^2w''w^2 \qquad \text{...eq(13)}$$

Substitute eqs (8), (12)and (13) into eq (11):

$$-h^2w''w^2 - \frac{1}{w}\left(\frac{h}{r^2}\right)^2 = -w^2GM \;\; or$$

$-h^2w''w^2 - \frac{1}{w}(h^2w^4) = -w^2GM$ Dividing through by $-h^2w^2$,

$$w'' + w = \frac{GM}{h^2}$$

$w(D^2 + 1) = \frac{GM}{h^2}$

A particular solution to the above differential equation is $w = A\cos\theta + \frac{GM}{h^2}$

But $w = 1/r = A\cos\theta + \frac{GM}{h^2}$

$r = \frac{1}{\frac{GM}{h^2} + A\cos\theta} = \frac{1}{\frac{GM}{h^2} + A\cos\theta}\left[\frac{\frac{h^2}{GM}}{\frac{h^2}{GM}}\right] = \frac{\frac{h^2}{GM}}{1 + A\frac{h^2}{GM}\cos\theta}$, which is in the form

$r = \frac{pe}{1 + e\cos\theta}$, *which is an ellipse.* ...Eq(14)