Derivation of Kepler's Third Law

Beginning with eq (7), $\frac{dA}{dt} = \frac{h}{2}$

$$dA = \frac{h}{2}dt$$

Integrating both sides, we can get an expression containing the total time of a planet's revolution around its sun, in other words, its period, T.

$$\int dA = \frac{h}{2}\int_0^T dt$$

$$A = \frac{h}{2}T \qquad \text{...eq (15)}$$

To relate the period T to the semi major axis, *a*, recall that we have shown that the path of revolution is elliptical. The area of an ellipse can be expressed as πab, where b = minor axis.

A= πab, ...eq (16)

Equating eqs (15) and (16):

$\frac{h}{2}T = \pi ab$ ...eq (17)

To get rid of the b term and introduce terms from the polar form of the ellipse, we need to convert the polar form into the cartesian one.

Recall eq (14):

$$r = \frac{pe}{1 + e\cos\theta}$$

Rearranging: pe = r + re$\cos\theta$

Or r = pe – re$\cos\theta$, ...eq(18)

but when converting from polar to cartesian coordinates, since x = $r\cos\theta$ and y = r$\sin\theta$, $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

Or $r = \sqrt{x^2 + y^2}$ ...eq(19)

Equating (18) and (19):

pe – re$\cos\theta$= $\sqrt{x^2 + y^2}$ ...squaring both sides...

$p^2e^2 - 2pre^2\cos\theta + r^2e^2cos^2\theta = x^2 + y^2$, but since $\cos\theta = \frac{x}{r}$

$p^2e^2 - 2pxe^2 + e^2x^2 = x^2 + y^2$

Rearranging and factoring:

$p^2e^2 = x^2(1-e^2) + 2pe^2x + y^2$. Dividing through by $(1-e^2)$:

$\frac{p^2e^2}{(1-e^2)} = x^2 + \frac{2pe^2}{(1-e^2)}x + \frac{y^2}{(1-e^2)}$. Completing the square for the x terms:

$\frac{p^2e^2}{(1-e^2)} + \frac{p^2e^4}{(1-e^2)^2} = \left[x + \frac{pe^2}{(1-e^2)}\right]^2 + \frac{y^2}{(1-e^2)}$. Simplifying the left side:

$\frac{p^2e^2}{(1-e^2)^2} = \left[x + \frac{pe^2}{(1-e^2)}\right]^2 + \frac{y^2}{(1-e^2)}$ and dividing through by $\frac{p^2e^2}{(1-e^2)^2}$:

$$1 = \frac{\left[x+\frac{pe^2}{(1-e^2)}\right]^2}{\frac{p^2e^2}{(1-e^2)^2}} + \frac{y^2}{(1-e^2)\frac{p^2e^2}{(1-e^2)^2}}$$ or...

$$1 = \frac{\left[x+\frac{pe^2}{(1-e^2)}\right]^2}{\left[\frac{pe}{1-e^2}\right]^2} + \frac{y^2}{\left[\frac{pe}{\sqrt{1-e^2}}\right]^2}$$ Comparing this to the general Cartesian form of the ellipse,

$1 = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$, we notice that $a = \frac{pe}{1-e^2}$ $and\ b = \frac{pe}{\sqrt{1-e^2}}$ ...Square rooting both sides of the "a" expression and solving for $\sqrt{1-e^2}, we\ get\ \sqrt{1-e^2} = \frac{\sqrt{pe}}{\sqrt{a}}$.

Substituting into the "b" expression:

b = $\frac{pe}{\frac{\sqrt{pe}}{\sqrt{a}}} = \sqrt{pea}$

Now we can substitute the above into equation (17) and obtain:

$\frac{h}{2}T = \pi a\sqrt{pea}$ or

$$T = \frac{2}{h}\pi\sqrt{pea^3}$$

Squaring both sides:

$$T^2 = \frac{4}{h^2}\pi^2 pea^3 \ . \qquad \text{eq (20)}$$

We already see Kelper's 3rd Law in that the square of the period is directly proportional to the cube of the semi major axis, but in deriving equation (14), one can notice that pe $= \frac{h^2}{GM}$. Substituting this into eq(20):

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

"I first believed I was dreaming… But it is absolutely certain and exact that the ratio which exists between the period times of any two planets is precisely the ratio of the 3/2th power of the mean distance."

- *Harmonies of the World* by Kepler (1619)